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Determination of Weighting Matrices of a Linear Quadratic Regulator

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Introduction

IN optimal control with a quadratic cost function, the selection of the state weighting matrix Q and the control weighting matrix R is normally based on an iterative procedure using experience and physical understanding of the problems involved. To find suitable Q and R that provide a desired balance between the state variable responses and control efforts while satisfying performance requirements and constraints, the transient response of a closed-loop system is typically examined. Because of indirect and nonlinear mapping between the weighting matrices and system closed-loop eigenvalues, it is difficult to find suitable Q and R. Certain general guidelines[1,2] are normally followed to construct Q and R; but these methods may not lead to satisfactory responses. The pole assignment methods[3,4] can provide certain types of connections between closed-loop poles (or eigenvalues) and feedback gains, and it tends to result in more accurate transient responses. By using pole assignment only without consideration of an optimal control cost function, however, it is difficult to balance state and control variables and to account for control effectiveness. The purpose of this Note is to present a systematic method for determining the weighting matrices Q and R to produce specified closed-loop eigenvalues. The method will be demonstrated through a numerical example.

Formulation

A linear system can be described by a state space equation

$$\dot{x} = Ax + Bu \tag{1}$$

where A is the state matrix of $n \times n$, and B is the control matrix of $n \times m$. Also, the pair (A, B) is assumed to be such that the system is controllable. In a linear quadratic regulator (LQR) problem, a cost function described by Eq. (2) needs to be optimized:

$$J = \int_0^\infty (x^T Qx + u^T Ru)\,\mathrm{d}t \tag{2}$$

where Q is a positive semidefinite state weighting matrix and R is a positive definite control weighting matrix. The problem can be formulated in terms of a Hamiltonian defined as

$$H = x^T Qx + u^T Ru + \lambda^T (Ax + Bu) \tag{3}$$

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where λ is the vector of Lagrangian multipliers. Then the solution can be obtained by solving the following equations:

$$\begin{aligned} \dot{\lambda} &= -\frac{\partial H}{\partial x} = -A^T\lambda - Qx, \qquad \lambda(\infty) = 0 \\ \dot{x} &= \frac{\partial H}{\partial \lambda} = Ax + Bu, \qquad x(0) = x_0 \\ \frac{\partial H}{\partial u} &= Ru + B^T\lambda = 0 \end{aligned} \tag{4}$$

which can be written as

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} = \tilde{A} \begin{bmatrix} x \\ \lambda \end{bmatrix} \tag{5}$$

where $\tilde{A}$ is a $2n \times 2n$ matrix, and n of its $2n$ eigenvalues are also the eigenvalues of the closed-loop system that satisfy

$$\det[\sigma I - \tilde{A}] = 0 \tag{6}$$

where σ represents the eigenvalues of $\tilde{A}$.

Instead of using an iterative procedure, the following method is proposed to determine the weighting matrices. The weighting matrix R will be chosen to have a diagonal form[1] having elements given by

$$R_{ii} = 1/u_{i,\max}^2, \qquad R_{ij} = 0, \quad i \neq j \tag{7}$$

to penalize each of the control input $u_1, u_2, \ldots, u_m$, and $u_{1,\max}$, $u_{2,\max}, \ldots, u_{m,\max}$ represent the maximum limits of each control input, respectively. The control weighting matrix cofficients can be collectively multiplied by a positive factor without altering the ratios between these coefficients. The weighting matrix Q is also assumed to have a diagonal form with elements given by

$$Q_{ii} = q_i, \qquad Q_{ij} = 0, \quad i \neq j \tag{8}$$

which are to be determined.

Equation (6) will be used to determine n elements $q_i (i = 1, 2, \ldots, n)$ of the weighting matrix Q if all closed-loop eigenvalues are specified. For a specified eigenvalue, $\sigma = \mu + i\omega$, Eq. (6) provides one equation for q_i

$$f(q_1, q_2, \ldots, q_n) = \det[(\mu + i\omega)I - \tilde{A}] = 0 \tag{9}$$

As a result, n algebraic equations,

$$F(s) = [f_1(s), f_2(s), \ldots, f_n(s)]^T = 0 \tag{10}$$

can be solved for the unknown vector $s = (q_1, q_2, \ldots, q_n)$. Expansion of high-order determinants, however, is a tedious task. A computer program[5] based on Chió's algorithm is employed as an alternative to evaluate the $2n$th-order determinant. Newton's method is then used to solve Eq. (10) for the vector s

$$s^{k+1} = s^k - \left[\frac{\partial F}{\partial s}\right]^{-1} \cdot F(s^k) \tag{11}$$

in which the Jacobian matrix, $\partial F/\partial s$ is calculated numerically at s^k. Since the matrix Q is supposed to be positive semidefinite, a negative q_i computed by the given method is replaced by zero, which means no penalty on the corresponding state variable. If several q_i are calculated to be negative, the control weighting matrix will need to be adjusted based on a specific control objective since the required maximum magnitudes of each control input for different objectives are different.

With the resulting weighting matrices Q and R, Ricatti's equation

$$-PA - A^T P + PBR^{-1}B^T P - Q = 0 \tag{12}$$

is used to find matrix P and, hence, the optimal feedback control

$$u = -R^{-1}B^T Px \tag{13}$$

Table 1 Eigenvalues in lateral-directional motions of a fighter aircraft

	Roll	Spiral	Dutch roll	Integrator
Open loop	−7.816	0.0076	$-0.757 \pm 5.806i$	N/A
Specified closed loop	−8.0	−0.05	$-4.88 \pm 3.66i$	−0.7
Resulting closed loop	−8.005	−0.0497	$-4.891 \pm 3.68i$	−0.6995

Numerical Example

To demonstrate this method, a linear model of a fighter aircraft is used. The aircraft is trimmed at Mach = 1.5 and $h = 10{,}000$ ft. The angle of attack $\alpha = 0.86$ deg, and the locally linearized lateral-directional equations of motion is

$$\begin{bmatrix}\dot\beta\\ \dot p\\ \dot r\\ \dot\phi\end{bmatrix} = \begin{bmatrix}-0.493 & 0.015 & -1.000 & 0.020\\ -61.176 & -7.835 & 4.991 & 0.000\\ 31.804 & -0.235 & -0.994 & 0.000\\ 0.000 & 1.000 & -0.015 & 0.000\end{bmatrix}\begin{bmatrix}\beta\\ p\\ r\\ \phi\end{bmatrix} + \begin{bmatrix}-0.002 & 0.002\\ 8.246 & 1.849\\ 0.249 & -0.436\\ 0.000 & 0.000\end{bmatrix}\begin{bmatrix}\delta_{df}\\ \delta_r\end{bmatrix} \tag{14}$$

where β, p, r, and ϕ represent the sideslip angle, pitch rate, yaw rate, and roll angle, whereas δ_{df} and δ_r are the deflection angles of differential flap and rudder, respectively.

For a zero steady-state error in sideslip, an integrator equation

$$\dot\xi_\beta = \beta - \beta_s \tag{15}$$

is added to Eq. (14), where β_s is the steady-state sideslip angle. The state and control vectors become

$$\boldsymbol{x} = (\beta, p, r, \phi, \xi_\beta)^T, \qquad \boldsymbol{u} = (\delta_{df}, \delta_r)^T$$

The elements of matrix R are chosen as

$$r_1 = 1/(\delta_{df,\max})^2 = 1/10^2, \qquad r_2 = 1/(\delta_{r,\max})^2 = 1/30^2$$

or $r_1 = 9$ and $r_2 = 1$ after multiplying by 900 ($= 30^2$), since only the relative magnitudes are needed.

If the specified closed-loop eigenvalues are −8.00 (roll), −0.05 (spiral), $-4.88 \pm 3.66i$ (dutch roll), and −0.7 (integrator), the elements of matrix Q are found from Eq. (10) to be

$$q_1 = -72.923, \qquad q_2 = 1.667, \qquad q_3 = 366.068$$
$$q_4 = 4.285, \qquad q_5 = 13.263$$

To ensure that Q is positive semidefinite, q_1 is set to 0. Solving the Ricatti equation (Eq. 12) with the computed weighting matrices Q and R, the full state control law can be determined as

$$\delta_r = 3.41(\beta - \beta_s) - 0.4126p + 15.947r - 0.865\phi - 3.536\int(\beta - \beta_s)\,\mathrm{d}t \tag{16}$$

$$\delta_{df} = -1.01(\beta - \beta_s) - 0.1576p - 0.8166r - 0.667\phi + 0.2904\int(\beta - \beta_s)\,\mathrm{d}t \tag{17}$$

As shown in Table 1, the resulting closed-loop eigenvalues are very close to the specified values.

Conclusions

A systematic method of determining weighting matrices for a linear quadratic regulator was proposed and studied. To obtain the specified closed-loop eigenvalues, the corresponding diagonal weighting matrices were numerically calculated. A tradeoff between penalties on the state and control inputs in the optimization of a cost function could also be analyzed. The computation normally needed less than 10 iterations and, thus, was very efficient.

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Recurrent Artificial Neural Network Simulation of a Chaotic System Without Training

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I. Introduction

At present, an area of increasing interest is the emulation and control of nonlinear dynamic systems by the recurrent artificial neural network (RANN) architecture. These particular networks use supervised learning algorithms (training algorithms) such as recurrent backpropagation,[1] which allow the network to simulate a dynamic system without knowledge of the governing equations. However, since supervised learning algorithms require exposure to numerous training data sets, it can become memory intensive and time consuming without the guarantee of success. As a result, RANNs are not yet reliable tools for engineering.

Before RANNs can be used with confidence in the engineering community, several questions must be answered such as how to guarantee convergence, how to increase accuracy, and how to match a particular connection scheme to an application. We believe these questions can be answered by starting with two basic assumptions that force a slight paradigm shift: 1) RANN "learning" is actually function approximation, and 2) RANNs can be manipulated as well as conventional numerical techniques. Consequently, RANNs can be constructed to perform tasks with accuracy and computational efficiency, and therefore are as applicable to hard computing as they are to soft computing.

The second assumption follows naturally from the first since all numerical techniques used in computational mechanics can be considered to be methods of function approximation. However, in the majority of cases the numerical techniques construct a function from an integral or differential equation rather than data sets. A number of researchers[2] have published work operating under the first assumption for artificial neural networks (ANNs) in general, but few[3,4] have investigated the second. To prove the latter assumption, one must fully investigate the parameters that govern RANN performance such as connection weights, activation functions, types of neurons (additive or multiplicative), and connection schemes. The most straightforward and logical approach to the development of the mathematics of RANNs is the solution of well-posed problems of increasing complexity.

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